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# ON APPROXIMATION OF AFFINE BAIRE-ONE FUNCTIONS*

BY

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ABSTRACT

It is known (G. Choquet, G. Mokobodzki) that a Baire-one affine function on a compact convex set satisfies the barycentric formula and can be expressed as a pointwise limit of a sequence of continuous affine functions. Moreover, the space of Baire-one affine functions is uniformly closed. The aim of this paper is to discuss to what extent analogous properties are true in the context of general function spaces.

In particular, we investigate the function space $H(U)$, consisting of the functions continuous on the closure of a bounded open set $U \subset \mathbb{R}^m$ and harmonic on $U$, which has been extensively studied in potential theory. We demonstrate that the barycentric formula does not hold for the space $\mathcal{B}_1^b(H(U))$ of bounded functions which are pointwise limits of functions from the space $H(U)$ and that $\mathcal{B}_1^b(H(U))$ is not uniformly closed. On the other hand, every Baire-one $H(U)$-affine function (in particular a solution of the generalized Dirichlet problem for continuous boundary data) is a pointwise limit of a bounded sequence of functions belonging to $H(U)$.

It turns out that such a situation always occurs for simplicial spaces whereas it is not the case for general function spaces. The paper provides several characterizations of those Baire-one functions which can be approximated pointwise by bounded sequences of elements of a given function space.

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## 1. Introduction

Our contribution follows a longstanding and fruitful symbiosis of abstract convex analysis and potential theory. A reader can consult the books H. Bauer [7], J. Bliedtner and W. Hansen [11], M. Brelot [13], G. Choquet [15], C. Constantinescu and A. Cornea [16], C. Dellacherie and P. A. Meyer [17] for introduction to the subject and further references.

Let $U$ be a bounded open subset of the Euclidean space $\mathbb{R}^m$ and $H(U)$ be the vector space of all continuous functions on the closure $\overline{U}$ of $U$ which are harmonic on $U$. Given a continuous function $f$ on the boundary $\partial U$ of $U$, define

$$f^{\complement U}\colon x \mapsto \int_{\partial U} f d\varepsilon_x^{\complement U} \quad \text{for each } x \in \overline{U}. \tag{1}$$

Here $\varepsilon_x^{\complement U}$ denotes the balayage of the Dirac measure $\varepsilon_x$ on the complement $\complement U$ of $U$, so that $\varepsilon_x^{\complement U}$ is the harmonic measure at $x$ for every $x \in U$. The restriction of $f^{\complement U}$ to $U$ is a harmonic function and it yields the solution $H^f$ of the generalized Dirichlet problem for the boundary condition $f$. It is known that, in the case of a non-regular set $U$, the function $f^{\complement U}$ need not be continuous on $\overline{U}$. However, $f^{\complement U}$ is a Baire-one function. Indeed, extending $f$ to the whole space as a continuous function with compact support, we can use the expression (1) to define $f^{\complement U}$ on $\mathbb{R}^m$. Then $f^{\complement U}$ is a finely continuous function by [16], Proposition 7.1.4, therefore a Baire-one function by [20]. (Recall that a real-valued function on a topological space $T$ is said to be a *Baire-one function* if it is a pointwise limit of a sequence of continuous functions on $T$. The set of all Baire-one functions on $T$ will be denoted by $\mathcal{B}_1(T)$.) Therefore, for every continuous function $f$ on $\partial U$, $f^{\complement U}$ is a pointwise limit of a sequence of functions continuous on $\overline{U}$. A natural question arises:

*Is it always possible to express $f^{\complement U}$ as a pointwise limit of a sequence of functions belonging to the function space $H(U)$ ?*

Consider now a more abstract framework of function spaces. A general background can be found, e.g., in E. M. Alfsen [1], L. Asimow and A. J. Ellis [4], R. R. Phelps [27].

Let $\mathcal{H}$ be a *function space* on a compact Hausdorff space $K$. By this we mean a (not necessarily closed) linear subspace of $\mathcal{C}(K)$ (the space of all real-valued continuous functions on $K$ equipped with the sup-norm) containing the constant functions and separating the points of $K$. We will identify the dual of $\mathcal{C}(K)$ with the space $\mathcal{M}(K)$ of all Radon measures on $K$. Let $\mathcal{M}^1(K)$ denote the set of all probability Radon measures on $K$. Then $\mathcal{M}^1(K)$ is a convex and $w^*$-compact subset of $\mathcal{M}(K)$. We denote by $\varepsilon_x$ the Dirac measure at $x \in K$.

Let $\mathcal{M}_x(\mathcal{H})$ be the set of all *$\mathcal{H}$-representing measures* for $x \in K$, i.e.,

$$\mathcal{M}_x(\mathcal{H}) := \{\mu \in \mathcal{M}^1(K)\colon f(x) = \int_K f d\mu \text{ for any } f \in \mathcal{H}\}.$$

The set

$$\mathrm{Ch}_{\mathcal{H}}(K) := \{x \in K\colon \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}$$

is called the *Choquet boundary* of $\mathcal{H}$. A point $x \in X$ is said to be an *exposed point* for $\mathcal{H}$ if there exists a function $f \in \mathcal{H}$ which attains a strict minimum (or a strict maximum) at $x$. It is easy to see that each exposed point belongs to the Choquet boundary.

We define the space $\mathcal{A}(\mathcal{H})$ of all *$\mathcal{H}$-affine functions* as the family of all bounded Borel functions on $K$ satisfying the following *barycentric formula:*

$$f(x) = \int_K f d\mu \quad \text{for each } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H}).$$

Further, let $\mathcal{A}^c(\mathcal{H})$ be the family of all continuous $\mathcal{H}$-affine functions on $K$ and

$$\mathcal{B}_1(\mathcal{H}) := \{f\colon \text{there is a sequence } \{f_n\} \text{ in } \mathcal{H} \text{ such that } f_n \to f \text{ on } K\}.$$

Of course, $f_n \to f$ on $K$ means the pointwise convergence, i.e., $f_n(x) \to f(x)$ whenever $x \in K$. Note that we have already introduced the notation $\mathcal{B}_1(K)$ for the family of all Baire-one functions on $K$. We see that $\mathcal{B}_1(K)$ is nothing else than $\mathcal{B}_1(\mathcal{C}(K))$. We shall denote by $\mathcal{B}_1^b(\mathcal{H})$ the family of all bounded elements of $\mathcal{B}_1(\mathcal{H})$ and by $\mathcal{B}_1^{bb}(\mathcal{H})$ the set of all functions on $K$ which are pointwise limits on $K$ of bounded sequences of functions from $\mathcal{H}$. As an immediate consequence of the Lebesgue dominated convergence theorem we obtain that

$$\mathcal{B}_1^{bb}(\mathcal{H}) \subset \mathcal{A}(\mathcal{H}). \tag{2}$$

In convex analysis, the role of pointwise convergence in connection with affinity is well understood. G. Choquet [14] proved that any Baire-one affine function on a compact convex set $X$ satisfies the barycentric formula. A theorem by G. Mokobodzki (see [29]) says that each Baire-one affine function on a compact convex set $X$ is a pointwise limit of continuous affine functions. See Section 4 for more information. We consider the situation in the framework of function spaces and particularly in the case of harmonic functions. Under some circumstances it is true that all bounded Baire-one functions from $\mathcal{A}(\mathcal{H})$ can be represented as pointwise limits of bounded sequences of functions from $\mathcal{H}$ or $\mathcal{A}^c(\mathcal{H})$. On the other hand, there are examples of limits of (now unbounded) sequences of functions from $\mathcal{A}^c(\mathcal{H})$ which are not $\mathcal{H}$-affine. All these phenomena are in the focus of our interest.

The plan of our paper is the following. Section 2 brings auxiliary results related to the Choquet theory of function spaces. Section 3 is devoted to potential theory. Ancona's theorem and a version of Bliedtner–Hansen's "smearing lemma" are applied there to show that $f^{\complement U} \in \mathcal{B}_1^b(H(U))$ whenever $f \in \mathcal{C}(\partial U)$. Several examples presented reveal an essential difference between the function space $H(U)$ and the space $A(X)$ of continuous affine functions on a compact convex set $X$. Namely, the barycentric formula need not hold for functions from $\mathcal{B}_1^b(H(U))$ and $\mathcal{B}_1^b(H(U))$ is not uniformly closed. Affine functions on compact convex sets are studied in Section 4. A slightly simplified proof of the Choquet theorem on Baire-one affine functions is given and the result is shown to lead to a maximum principle for Baire-one affine functions. Section 5 provides a characterization of those Baire-one functions which are pointwise limits of a bounded sequence of elements from a function space in question. An example shows that, for a suitable function space $\mathcal{H}$, Baire-one $\mathcal{H}$-affine functions need not coincide with $\mathcal{B}_1(\mathcal{H})$. This cannot occur in simplicial spaces, as proved in the final Section 6. This offers still another proof of the fact that $f^{\complement U} \in \mathcal{B}_1^b(H(U))$ for every $f \in \mathcal{C}(\partial U)$. For simplicial spaces, we also construct a sequence of positive linear operators producing, for each continuous function, an approximation of a solution to the abstract Dirichlet problem by means of continuous $\mathcal{H}$-affine functions.

Before proceeding, some notational conventions will be established.

We use *positive* for $\geq 0$ and *strictly positive* for $> 0$.

For a linear space $\mathcal{E}$ of bounded Borel functions on a compact space $K$, we will denote by $\|\cdot\|$ the sup-norm on $\mathcal{E}$. The annihilator $\mathcal{F}^\perp$ of a family $\mathcal{F} \subset \mathcal{E}$ of bounded Borel functions on $K$ is defined by

$$\mathcal{F}^\perp := \{\mu \in \mathcal{M}(K) \colon \mu(f) = 0 \text{ for any } f \in \mathcal{F}\}. \tag{3}$$

Note that annihilators (3) are here always taken as subsets of spaces of Radon measures. This agrees with the general notion of an annihilator if we consider the duality between $\mathcal{E}$ and $\mathcal{M}(K)$ rather than the duality between $\mathcal{E}$ and $(\mathcal{E}, \|\cdot\|)^*$.

For $\mu \in \mathcal{M}(K)$, $\operatorname{spt}\mu$ stands for its support. Given a $\mu$-integrable function $\varphi$, the Radon measure having the density $\varphi$ with respect to $\mu$ is labelled as $\varphi\mu$. For integrable functions on $K$, we simply write $\mu(f)$ instead of $\int_K f\,d\mu$.

Let $X$ be a compact convex subset of a real locally convex space. Recall that a real function $f$ (not necessarily continuous) is said to be *affine* on $X$ if $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$ for each $x, y \in X$ and for each $\lambda \in (0,1)$.

Denote by $\mathfrak{A}(X)$ the set of all affine functions on $X$. Let $A(X) := \mathfrak{A}(X) \cap \mathcal{C}(X)$ be the set of all affine continuous functions on $X$.

We denote by $B(x, r)$ the open ball with center at $x$ and radius $r$ in the Euclidean space $\mathbb{R}^m$.

All topological spaces will be considered as Hausdorff.

## 2. Function spaces

In the sequel we consider the following main examples of function spaces. In the "convex case" the function space $\mathcal{H}$ is the linear space $A(X)$ of all continuous affine functions on a compact convex subset $X$ of a locally convex space. In the "harmonic case" $U$ is a bounded open subset of the Euclidean space $\mathbb{R}^m$ and the corresponding function space $\mathcal{H}$ is $H(U)$, i.e., the family of all continuous functions on $\overline{U}$ which are harmonic on $U$.

Before going further we present results needed in the next sections. First, we come to a simple but very important well-known folklore result which is useful for a general context of function spaces. So let $\mathcal{H}$ be a function space on a compact space $K$ and let $f$ be an upper bounded function on $K$. For $x \in K$ put

$$f^*(x) := \inf\{h(x) \colon h \in \mathcal{H}, h \geq f \text{ on } K\}.$$

Obviously, $f^*$ is an upper semicontinuous function on $K$. Similarly, for a lower bounded function $f$ on $K$, we define $f_*$ so that $f_*(x) = -(-f)^*(x)$, $x \in K$.

For the case of continuous affine functions, the proof of the next lemma can be found in [1], Corollary I.3.6 (cf. also Remark following this Corollary). We give a self-contained proof.

LEMMA 2.1: *Let $\mathcal{H}$ be a function space on $K$, $f$ be an upper semicontinuous function on $K$ and $x \in K$. Then there exists $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $f^*(x) = \mu(f)$.*

*Proof:* Fix an $x$ in $K$ and assume first that $f \in \mathcal{C}(K)$. The mapping $p\colon g \mapsto g^*(x)$ is a sublinear functional on $\mathcal{C}(K)$. The Hahn–Banach theorem provides a linear functional $\mu_f$ on $\mathcal{C}(K)$ such that $\mu_f(f) = f^*(x)$ and $\mu_f(g) \leq g^*(x)$ for any $g \in \mathcal{C}(K)$. Since $p(g) \leq 0$ whenever $g \in \mathcal{C}(K)$ and $g \leq 0$, we see that $\mu_f$ is a positive Radon measure on $K$. Let $h \in \mathcal{H}$. Then

$$\mu_f(h) \leq p(h) = h^*(x) = h(x)$$

and simultaneously

$$-\mu_f(h) = \mu_f(-h) \leq p(-h) = (-h)^*(x) = -h_*(x) = -h(x).$$

We see that $\mu_f \in \mathcal{M}_x(\mathcal{H})$.

Let now $f$ be an upper semicontinuous function on $K$. Denote by $\mathcal{G}$ the lower directed set $\{g \in \mathcal{C}(K)\colon g \geq f \text{ on } K\}$. For any $g \in \mathcal{G}$ there is a measure $\mu_g \in \mathcal{M}_x(\mathcal{H})$ such that $\mu_g(g) = g^*(x)$. Given $\varphi \in \mathcal{G}$, let

$$M_\varphi = \{\mu_g\colon g \in \mathcal{G}, g \leq \varphi\}.$$

By a compactness argument, there is $\mu \in \bigcap_{\varphi \in \mathcal{G}} \overline{M_\varphi}^{w^*}$. A moment's reflection shows that $\mu \in \mathcal{M}_x(\mathcal{H})$. We observe that

$$\inf\{\nu(\varphi)\colon \nu \in M_\varphi\} = \inf\{\nu(\varphi)\colon \nu \in \overline{M_\varphi}^{w^*}\} \leq \mu(\varphi)$$

for each $\varphi \in \mathcal{G}$. Hence

$$\begin{aligned} f^*(x) \leq& \inf\{g^*(x)\colon g \in \mathcal{G}\} = \inf\{\mu_g(g)\colon g \in \mathcal{G}\} \\ \leq& \inf\{\inf\{\mu_g(\varphi)\colon g \in \mathcal{G}, g \leq \varphi\}\colon\ \varphi \in \mathcal{G}\} \leq \inf\{\mu(\varphi)\colon\ \varphi \in \mathcal{G}\} \\ =& \mu(f) \leq \inf\{\mu(h)\colon h \geq f, h \in \mathcal{H}\} = \inf\{h(x)\colon h \geq f, h \in \mathcal{H}\} \\ =& f^*(x), \end{aligned}$$

which are the inequalities needed to finish the proof. ∎

LEMMA 2.2: *Let $f$ be a bounded function on $K$. Then*

$$f^*(x) = \inf\{g(x)\colon g \in \mathcal{A}^c(\mathcal{H}),\ g \geq f \text{ on } K\}, \quad x \in K.$$

*Proof:* Recall that

$$f^*(x) := \inf\{h(x)\colon h \in \mathcal{H},\ h \geq f \text{ on } K\}, \quad x \in K.$$

Given $x \in K$ and $g \in \mathcal{A}^c(\mathcal{H})$, $g \geq f$, by Lemma 2.1 there is a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ such that $g^*(x) = \mu(g)$. Then

$$\begin{aligned} g(x) =& \mu(g) = g^*(x) \geq f^*(x) = \inf\{h(x)\colon h \in \mathcal{H},\ h \geq f\} \\ \geq& \inf\{\tilde{g}(x)\colon \tilde{g} \in \mathcal{A}^c(\mathcal{H}),\ \tilde{g} \geq f\}. \end{aligned}$$

Taking the infimum over all $g$ in $\mathcal{A}^c(\mathcal{H})$ finishes the reasoning. ∎

A bounded Borel function $f$ on $K$ is called *$\mathcal{H}$-convex* if $f(x) \leq \mu(f)$ for any $x \in K$ and any $\mu \in \mathcal{M}_x(\mathcal{H})$. Let $\mathcal{K}^c(\mathcal{H})$ be the family of all continuous $\mathcal{H}$-convex functions on $K$. This convex cone determines a partial ordering on the space of all positive Radon measures on $K$: $\mu \prec \nu$ if $\mu(f) \leq \nu(f)$ for each $f \in \mathcal{K}^c(\mathcal{H})$. One of the most important results of the Choquet theory says that for each $x \in K$

there always exists a maximal measure in $\mathcal{M}_x(\mathcal{H})$ (with respect to the ordering $\prec$).

The following result due to G. Mokobodzki (cf. [1], Proposition I.5.9) characterizes maximal measures:

MOKOBODZKI'S MAXIMALITY TEST: *A positive Radon measure $\mu$ on $K$ is maximal if and only if $\mu(k) = \mu(k^*)$ for any $k \in \mathcal{K}^c(\mathcal{H})$.*

Now, a function space $\mathcal{H}$ is called *simplicial* if for each $x \in K$ there is a unique maximal measure $\delta_x \in \mathcal{M}_x(\mathcal{H})$. In the convex case $\mathcal{H} = A(X)$ we say simply that $X$ is a *Choquet simplex.*

We state here a well known "in-between property" for simplicial spaces, which in fact characterizes simpliciality.

EDWARDS SEPARATION THEOREM: *Let a function space $\mathcal{H}$ on $K$ be simplicial. Let $-f, g \in \mathcal{K}^c(\mathcal{H})$, $g \leq f$. Then there is $h \in \mathcal{A}^c(\mathcal{H})$ with $g \leq h \leq f$ on $K$.*

*Proof:* See [18], Theorem 3, cf. also [12], Theorem 3.2. ∎

In the following proofs we need a consequence of the Edwards separation theorem.

LEMMA 2.3: *If a function space $\mathcal{H}$ on $K$ is simplicial and $\delta_x$ is a unique maximal measure in $\mathcal{M}_x(\mathcal{H})$, then*

$$\delta_x(g) = \delta_x(g^*) = g^*(x)$$

*for any $g \in \mathcal{K}^c(\mathcal{H})$.*

*Proof:* Fix $g \in \mathcal{K}^c(\mathcal{H})$. The first equality follows from the above Mokobodzki test. Now, for each $h \in A^c(\mathcal{H})$ we have $\delta_x(h) = h(x)$ and, according to the Edwards separation theorem, the family $\{h \in \mathcal{A}^c(\mathcal{H})\colon h \geq g\}$ is lower directed. Therefore, using Lemma 2.2

$$\begin{aligned} g^*(x) &= \inf\{h(x)\colon h \in \mathcal{A}^c(\mathcal{H}), h \geq g\} = \inf\{\delta_x(h)\colon h \in \mathcal{A}^c(\mathcal{H}), h \geq g\} \\ &= \delta_x(g^*), \end{aligned}$$

and we arrive at the second equality. ∎

At the end of this section we shall recall the notion of the state space and summarize its basic properties. This notion represents a natural and efficient link between function spaces and convex analysis. Let $\mathcal{H}$ be a function space on a compact space $K$ and let $\mathbf{S}(\mathcal{H})$ denote the *state space* of $\mathcal{H}$ defined as

$$\mathbf{S}(\mathcal{H}) := \{\varphi \in \mathcal{H}^*\colon \varphi \geq 0, \varphi(1) = 1\}.$$

Clearly, $\mathbf{S}(\mathcal{H})$ is a convex $w^*$-compact subset of the dual space $\mathcal{H}^*$.

It is well known that $\mathcal{H}^*$ can be identified with the quotient space

$$(\mathcal{M}(K), w^*)/\mathcal{H}^\perp$$

equipped with the quotient (locally convex) topology.

We denote by $\pi$ the quotient mapping from $\mathcal{M}(K)$ onto $\mathcal{H}^*$. As a simple consequence of the Hahn–Banach theorem we obtain

$$\mathbf{S}(\mathcal{H}) = \pi(\mathcal{M}^1(K)). \tag{4}$$

Let $\phi$: $K \to \mathbf{S}(\mathcal{H})$ be the evaluation mapping defined as $\phi(x) = s_x$, $x \in K$ where $s_x(h) = h(x)$ for $h \in \mathcal{H}$. Obviously $\phi(x) = \pi(\varepsilon_x)$.

Let $\Phi$: $\mathcal{H} \to A(\mathbf{S}(\mathcal{H}))$ be the mapping defined for $h \in \mathcal{H}$ by $\Phi(h)(s) := s(h)$, $s \in \mathbf{S}(\mathcal{H})$. It is known (cf. [1], p. 80) that $\Phi$ serves as an isometric isomorphism of $\mathcal{H}$ into $A(\mathbf{S}(\mathcal{H}))$, and $\Phi$ is onto if and only if the function space $\mathcal{H}$ is uniformly closed in $\mathcal{C}(K)$. In this case the inverse mapping is realized by

$$\Phi^{-1}(F) = F \circ \phi, \quad F \in A(\mathbf{S}(\mathcal{H})). \tag{5}$$

We call a bounded Borel function $f$ on $K$ *completely $\mathcal{H}$-affine* if $\mu(f) = 0$ for each $\mu \in \mathcal{H}^\perp$. The set of all completely $\mathcal{H}$-affine bounded Borel functions on $K$ will be denoted by $\mathbf{A}(\mathcal{H})$. (These functions are termed *"fonctions qui vérifient la calcul barycentrique modulo $\mathcal{H}$"* by M. Rogalski in [29].)

A continuous function $h$ is completely $\mathcal{H}$-affine if and only if $h \in \overline{\mathcal{H}}$. This is an easy consequence of the Hahn–Banach theorem.

## 3. On the space $\mathcal{B}_1^b(H(U))$

Throughout $U$ is a bounded open subset of $\mathbb{R}^m$, $m \geq 2$. We will give different arguments proving that $f^{\complement U} \in \mathcal{B}_1^b(H(U))$ whenever $f \in \mathcal{C}(\partial U)$. We also exhibit several examples illustrating the character of functions belonging to $\mathcal{B}_1^b(H(U))$ and showing a dramatic difference between the nature of functions from $\mathcal{B}_1^b(H(U))$ and $\mathcal{B}_1^b(A(X))$.

We will use the standard potential theoretic notation; cf. [3], [11]. Let $\Omega = \mathbb{R}^m$, if $m > 2$, and let $\Omega$ be an open disc containing $\overline{U}$, if $m = 2$. The term *potential on* $\Omega$ means a Newtonian potential, if $m > 2$, and a Green potential with respect to $\Omega$, if $m = 2$. The symbol cap stands for the corresponding capacity on $\Omega$. Let $\mathcal{P}^c$ be the family of all continuous potentials on $\Omega$. Where no confusion can result, we will omit the notation for the restrictions $p \restriction \partial U$ or $p \restriction \overline{U}$.

For $A \subset \Omega$, $x \in \Omega$ and $p \in \mathcal{P}^c$, denote by $\varepsilon_x^A$ the *balayage* of $\varepsilon_x$ on $A$, $b(A)$ the *base* of $A$, i.e., the set of all points at which $A$ is not thin, and $\widehat{R}_p^A$ the *balayage* of $p$ on $A$. So $\widehat{R}_p^A(x) = \varepsilon_x^A(p)$, whenever $p \in \mathcal{P}^c$.

Let us recall that, for any $x \in \overline{U}$, the measure $\varepsilon_x^{\complement U}$ belongs to $\mathcal{M}_x(H(U))$. This immediately follows, e.g., from [11], Corollary VI.11.6.

Next we introduce the concept of an $H$-sequence. A positive linear operator $T$ from $\mathcal{C}(\partial U)$ into $H(U)$ is said to be an *$H$-operator on $U$* if $T1 \leq 1$. A sequence $\{T_n\}$ of $H$-operators will be called an *$H$-sequence on $U$* if $T_n f \to f^{\complement U}$ pointwise on $\overline{U}$ whenever $f \in \mathcal{C}(\partial U)$. With these preliminaries we may state the following result.

PROPOSITION 3.1: *Let $\{T_n\}$ be a sequence of $H$-operators. If $T_n p(x) \to \widehat{R}_p^{\complement U}(x)$ for every $p \in \mathcal{P}^c$ and every $x \in \overline{U}$, then $\{T_n\}$ is an $H$-sequence.*

*Proof:* For each $x \in \overline{U}$ and each $n$, let $\tau_n^x$ be a positive Radon measure on $\partial U$ such that $T_n f(x) = \int_{\partial U} f \, d\tau_n^x$, $f \in \mathcal{C}(\partial U)$. To complete the proof we only have to show that $\tau_n^x \overset{w^*}{\to} \varepsilon_x^{\complement U}$ as $n \to \infty$ for any $x \in \overline{U}$. We have $\tau_n^x(p) = T_n p(x) \to \widehat{R}_p^{\complement U}(x) = \varepsilon_x^{\complement U}(p)$ whenever $p \in \mathcal{P}^c$. Since the sequence $\{\tau_n^x\}_n$ is bounded and $(\mathcal{P}^c - \mathcal{P}^c) \restriction \partial U$ is dense in $\mathcal{C}(\partial U)$ by [16], Theorem 2.3.1, we conclude that $\tau_n^x \overset{w^*}{\to} \varepsilon_x^{\complement U}$. ∎

We give two proofs of the main result of this section, Theorem 3.2, making an essential use either of an important result of Ancona, or of a simplified version of the nice result of Bliedtner and Hansen on a "smearing of balayage". Still another proof of the result uses simpliciality of the function space $H(U)$ and is presented below, Remark 6.5.

ANCONA'S THEOREM: *Every compact, nonpolar set $K \subset \mathbb{R}^m$ contains a compact set $K' \neq \emptyset$ such that $K'$ is not thin at any of its points. Moreover, for each $\varepsilon > 0$, $K'$ can be chosen in such a way that $\operatorname{cap}(K \setminus K') < \varepsilon$.*

*Proof:* See [2]. ∎

BLIEDTNER–HANSEN'S LEMMA: *There exists a family $\{K_t\}_{0<t<1}$ of compact subsets of $\partial U \cap b(\complement U)$ such that $s < t \Longrightarrow K_s \subset K_t$, $\partial U \cap b(\complement U) = b(\bigcup_{0<t<1} K_t)$ and $K_s \subset b(K_t)$ for all $0 < s < t < 1$.*

*Proof:* See [11], Theorem VI.6.13. ∎

THEOREM 3.2: *There exists an $H$-sequence $\{T_n\}$ on $U$ such that the inequality $T_n(p) \le p$ holds for each $p \in \mathcal{P}^c$. Consequently, $f^{\complement U} \in \mathcal{B}_1^{bb}(H(U))$ whenever $f \in \mathcal{C}(\partial U)$.*

*First proof:* Ancona's theorem provides an increasing sequence $\{K_n\}$ of compact subsets of $\partial U$ such that each $K_n$ is not thin at any of its points and $\operatorname{cap}(\partial U \setminus K_n) < 1/n$. Denote $K := \bigcup_n K_n$. Then $\operatorname{cap}(\partial U \setminus K) = 0$ and for every $p \in \mathcal{P}^c$

$$\sup \widehat{R}_p^{K_n} = \widehat{R}_p^K \quad \text{on } \overline{U}.$$

Our aim is to show that $\widehat{R}_p^K = \widehat{R}_p^{\partial U} = \widehat{R}_p^{\complement U}$. But this follows from [16], Corollary 6.2.1, since the set $\partial U \setminus K$ is of capacity zero, and therefore polar.

If $f \in \mathcal{C}(\partial U)$, define $T_n f$ as $T_n f(x) = \varepsilon_x^{K_n}(f)$, $x \in \overline{U}$. Since the compact set $K_n$ is not thin at any of its points and $f \in \mathcal{C}(K)$, $T_n f \in H(U)$ (cf. [16], Proposition 7.1.4 or [11], Proposition VI.2.10). We have $T_n p \le \widehat{R}_p^{\complement U} \le p$ for every $p \in \mathcal{P}^c$. Since $T_n p(x) = \widehat{R}_p^{K_n}(x) \nearrow \widehat{R}_p^{\complement U}(x)$ for every $p \in \mathcal{P}^c$ and $x \in \overline{U}$, in light of Proposition 3.1 we conclude that $\{T_n\}$ is an $H$-sequence on $U$, and therewith the theorem is established. ∎

*Second proof:* Let $\{K_t\}_{0<t<1}$ be as furnished by Bliedtner–Hansen's lemma. For $x \in \overline{U}$ and $n \in \mathbb{N}$ define the *smearing of balayage*

$$T_n f(x) := 2^n \int_{1-2^{-n+1}}^{1-2^{-n}} \varepsilon_x^{K_t}(f)dt, \quad f \in \mathcal{C}(\partial U).$$

By [11], p. 314, p. 298, for every $p \in \mathcal{P}^c$, the function $T_n p$ belongs to $H(U)$ and $T_n p \nearrow \widehat{R}_p^{\complement U} \le p$ on $\overline{U}$.

In order to show that $\{T_n\}$ is an $H$-sequence on $U$, it remains to prove that $T_n f \in H(U)$ whenever $f \in \mathcal{C}(\partial U)$. It is sufficient to consider a positive $f \in \mathcal{C}(\partial U)$. Let $p_0 \in \mathcal{P}^c$, $p_0 > 0$ on $\Omega$ and $\varepsilon > 0$. There exist $p, q \in \mathcal{P}^c$ such that $|f - (p-q)| \le \varepsilon p_0$ on $\partial U$ (cf. [16], Theorem 2.3.1). This yields, for every $n \in \mathbb{N}$ and $x \in \overline{U}$, $|T_n f(x) - (T_n p(x) - T_n q(x))| \le \varepsilon \sup p_0(\overline{U})$. Since the space $H(U)$ is uniformly closed and $T_n p, T_n q \in H(U)$, it follows that $T_n f \in H(U)$. ∎

*Remark 3.3:* The second proof of Theorem 3.2 is applicable in a more general context such as that of a strong harmonic space where the essential base of $\complement U$ (cf. [11]) coincides with $b(\complement U)$.

As a consequence of the previous theorem we get the following assertion concerning the simpliciality of the space $H(U)$. Note that there are more elementary proofs in the case of potential theory on $\mathbb{R}^m$ ([19], Corollary 4.3 or [25]) and that there are more general and deeper results in abstract potential theory (cf. [10], Corollary 3.8).

PROPOSITION 3.4: *The function space $H(U)$ is simplicial. For any $x \in \overline{U}$ the balayage $\varepsilon_x^{\complement U}$ is a unique maximal measure in $\mathcal{M}_x(H(U))$.*

*Proof:* Let $x \in \overline{U}$ and let $\delta_x$ be a maximal measure in $\mathcal{M}_x(H(U))$. If $w \in \mathcal{K}^c(H(U))$, then, since $\varepsilon_y^{\complement U}$ is a representing measure for $y$ for any $y \in \overline{U}$, $w^{\complement U} \geq w$. Further, the function $w^{\complement U}$ is $H(U)$-affine by Theorem 3.2 and (2). Hence

$$\varepsilon_x^{\complement U}(w) = w^{\complement U}(x) = \delta_x(w^{\complement U}) \geq \delta_x(w).$$

It follows that $\varepsilon_x^{\complement U} \succ \delta_x$. Since $\delta_x$ is maximal, we conclude that $\delta_x = \varepsilon_x^{\complement U}$. ∎

From Corollary 6.4 it will follow that $\mathcal{B}_1^{bb}(H(U)) = \mathcal{B}_1^b(\overline{U}) \cap \mathcal{A}(H(U))$. However, $\mathcal{B}_1^b(H(U)) \neq \mathcal{B}_1^{bb}(H(U))$, as shown in the following examples. This means also that the barycentric formula can fail for functions from $\mathcal{B}_1^b(H(U))$. We will also demonstrate that the space $\mathcal{B}_1^b(H(U))$ is not closed with respect to uniform convergence. Analogous features cannot occur in the function space $A(X)$ of affine continuous functions on a compact convex set $X$ because in this case, by the Mokobodzki approximation theorem (see Section 4), $\mathcal{B}_1^b(X) \cap \mathcal{A}(A(X)) = \mathcal{B}_1^b(A(X)) = \mathcal{B}_1^{bb}(A(X))$.

LEMMA 3.5: *Suppose that $0 < r < R$. Let $f$ be a Baire-one function on $\mathbb{R}^m$ such that $f$ is harmonic on $\{x \in \mathbb{R}^m : |x| \notin \{0, r, R\}\}$. Then there is a sequence $\{h_n\}$ of harmonic functions on $\mathbb{R}^m$ such that $h_n(x) \to f(x)$ for each $x \in \mathbb{R}^m$.*

*Proof:* Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}^m$ such that $f_n \to f$. For each $n \in \mathbb{N}$ we choose numbers $\delta_n$, $\rho_n$, $r_n$, $r_n'$, $R_n$, $R_n'$ and $\kappa_n$ such that

$$0 < \delta_n < \rho_n < r_n < r < r_n' < R_n < R < R_n' < \kappa_n$$

and

$$\delta_n, \rho_n \searrow 0, \quad r_n \nearrow r, \quad r_n' \searrow r, \quad R_n \nearrow R, \quad R_n' \searrow R, \quad \kappa_n \nearrow \infty.$$

Set

$$\begin{gathered} K_n^1 = \{x \in \mathbb{R}^m \colon |x| \in [\rho_n, r_n] \cup [r_n', R_n] \cup [R_n', \kappa_n]\}, \\ K^2 = \{x \in \mathbb{R}^m \colon |x| \in \{0, r, R\}\}, \\ P_n = \{x = (x_1, \dots, x_m) \in \mathbb{R}^m \colon 0 < x_1 < \delta_n\}, \\ K_n = (K_n^1 \cup K^2) \setminus P_n. \end{gathered}$$

Let $g_n$ be a function on $K_n$ which is equal to $f$ on $K_n \cap K_n^1$ and to $f_n$ on $K_n \cap K^2$. Notice that the set $\mathbb{R}^m \setminus K_n$ is connected, the function $g_n$ is continuous on the compact set $K_n$ and harmonic on its interior $K_n^\circ$, and that $b(\mathbb{R}^m \setminus K_n) = b(\mathbb{R}^m \setminus K_n^\circ)$. By Mergelyan's type theorem [21], Theorem 1.15, there is a harmonic function $h_n$ on $\mathbb{R}^m$ such that $|h_n - g_n| < 1/n$ on $K_n$. Since $\bigcup_n K_n = \mathbb{R}^m$, it follows that $h_n \to f$ as $n \to \infty$. ∎

LEMMA 3.6: *Let $B = B(0,1)$ be the open unit ball in $\mathbb{R}^m$ and let $\{B_j\}$ be a sequence of pairwise disjoint balls in $B$, $B_j = B(x_j, r_j)$, such that the union $V$ of all balls $B_j$ is dense in $B$. Let $U \subset B$ be an open set containing $V$ and $f \in \mathcal{B}_1(H(U))$. If $f = 0$ on $\overline{U} \setminus V$, then there is $k \in \mathbb{N}$ such that $f(x_k) = 0$.*

*Proof:* Let us assume that $f(x_j) \neq 0$ for each $j \in \mathbb{N}$. Let $h_n$ be a sequence of functions from $H(U)$ such that $h_n \to f$. Let $x \in K := \overline{U} \setminus V$ and $G$ be an open set containing $x$. Then there is $k \in \mathbb{N}$ such that $\overline{B}_k \subset G$. Let $\sigma_k$ be the surface measure on $\partial B_k$ normalized by $\sigma_k(\partial B_k) = 1$. Then, by the mean value property of harmonic functions, $\sigma_k(h_n) \to f(x_k) \neq 0$. If we take into account that $h_n \to 0$ on $\partial B_k$, by the Lebesgue dominated convergence theorem the sequence $\{h_n\}$ cannot be bounded on $\partial B_k$. In particular it follows that

$$K \cap G \cap \bigcup_{i \geq n} \{|h_i| > 1\} \supset \partial B_k \cap \bigcup_{i \geq n} \{|h_i| > 1\} \neq \emptyset, \qquad n \in \mathbb{N}.$$

Appealing to the Baire category theorem we deduce that

$$\bigcap_{n \in \mathbb{N}} \bigcup_{i \geq n} (K \cap \{|h_i| > 1\}) \neq \emptyset,$$

which is a contradiction, because $h_n \to 0$ on $K$. ∎

*Example 3.7:* Let $U \subset \mathbb{R}^m$ be a bounded open set. Then $\chi_{\{0\}}$, the characteristic function of $\{0\}$, belongs to $\mathcal{B}_1^b(H(U))$.

*Proof:* This is an immediate consequence of Lemma 3.5. ∎

*Example 3.8:* Let $U \subset \mathbb{R}^m$ be a bounded open set and $B \subset U$ be an open ball. Then there is a function $h \in \mathcal{C}(\overline{U}) \cap \mathcal{B}_1^b(H(U))$ such that $h = 0$ on $\overline{U} \setminus B$ and $h > 0$ on $B$.

*Proof:* Without loss of generality we can assume that $B = B(0, R)$. Choose $r \in (0, R)$. Let $h$ be a continuous function on $\mathbb{R}^m$ which vanishes outside $B(0, R)$, equals 1 on $\overline{B}(0,r)$ and solves the Dirichlet problem on $B(0,R)\setminus\overline{B}(0,r)$ with boundary values 1 on $\partial B(0,r)$ and 0 on $\partial B(0,R)$. By Lemma 3.5, such a function $h$ belongs to $\mathcal{B}_1(H(U))$. ∎

*Remark 3.9:* Let $U\subset\mathbb{R}^m$ be a bounded open set and $B=B(0,R)\subset U$. Let $f=\chi_{\{0\}}$. By Example 3.7, $f\in\mathcal{B}_1^b(H(U))$. Obviously $f(0)\neq\varepsilon_0^{\complement U}(f)=0$, so that the "barycentric formula" fails. Also, for the function $h$ from Example 3.8 the "barycentric formula" fails for the same reason. We see that a function from $\mathcal{B}_1^b(H(U))$ need not be continuous in $U$, and even if it is continuous, the harmonicity on $U$ may be violated.

*Remark 3.10:* One may ask how far a function from the class $\mathcal{B}_1^b(H(U))$ may be from being harmonic. A harmonic version of a result of Osgood [26] says that each $h\in\mathcal{B}_1(H(U))$ must be harmonic on a dense open subset of $V$. The proof of this is a quite standard application of the Baire category theorem and the fact that a locally bounded sequence of harmonic functions on an open set converges locally uniformly provided it converges pointwise.

*Example 3.11:* Suppose that $U$ is a bounded open subset of $\mathbb{R}^m$. Then there exists a function $f$ from $\mathcal{C}(\overline{U})$ which is harmonic on a dense open subset of $U$ but does not belong to $\mathcal{B}_1^b(H(U))$.

*Proof:* We may suppose that the unit ball $B$ is contained in $U$. Let $\{B_j\}$ be as in Lemma 3.6. For each $k\in\mathbb{N}$ let $f_k$ be a function from $\mathcal{C}(\overline{U})\cap\mathcal{B}_1^b(H(U))$ such that $0<f_k\leq 1$ on $B_k$ and $f_k=0$ on $\overline{U}\setminus B_k$. The existence of such a function follows from Example 3.8. By Lemma 3.6, the function

$$f=\sum_{k=1}^{\infty}2^{-k}f_k$$

does not belong to $\mathcal{B}_1^b(H(U))$. We easily observe that $f$ is continuous on $\overline{U}$ and harmonic on a dense open subset of $U$. ∎

*Remark 3.12:* Lemma 3.5 yields a complete description of the family of all functions from $\mathcal{B}_1^b(H(U))$ which are harmonic on $U$, if $U$ is a ball. Namely, a bounded function on $\overline{U}$ which is harmonic on $U$ belongs to $\mathcal{B}_1^b(H(U))$ if and only if it belongs to $\mathcal{B}_1^b(\overline{U})$. However, the structure of the family of those functions from $\mathcal{B}_1^b(H(U))$ which are not harmonic on $U$ is much more complicated, as illustrated by Remark 3.10 and Example 3.11. Related questions for holomorphic functions were studied by Hartogs and Rosenthal [22]; for a new treatment including the harmonic case we refer to Štěpničková [31].

The family $\mathcal{B}_1(P)$ of all Baire-one functions on a metric space $P$ is closed with respect to uniform convergence (cf. [24], Lemma 3.5). More generally, if $\Phi$ denotes a vector lattice of real functions on a set $Y$ and $\Phi$ contains the constant functions, then the family $\mathcal{B}_1(\Phi)$ is likewise closed under uniform convergence (cf. [24], 3.G.2). Further, if $X$ is a compact convex set, then $\mathcal{B}_1(A(X)) = \mathfrak{A}(X) \cap \mathcal{B}_1(X)$ by Mokobodzki's approximation theorem stated in Section 4. Therefore the space $\mathcal{B}_1(A(X))$ is also closed under uniform convergence.

Therefore a natural question arises:

*Is the space $\mathcal{B}_1^b(H(U))$ uniformly closed?*

We are going to present examples giving a negative answer to this question.

EXAMPLE 3.13: *Suppose that $U$ is a bounded open subset of $\mathbb{R}^m$. Then the space $\mathcal{B}_1^b(H(U))$ is not uniformly closed.*

*Proof:* Let $Z := \{z_j : j \in \mathbb{N}\}$ be a countable dense subset of $U$. By Example 3.7, $\chi_{\{z_j\}} \in \mathcal{B}_1^b(H(U))$. Set

$$h_k = \sum_{j=1}^{k} 2^{-j} \chi_{\{z_j\}} \quad \text{on } \overline{U}.$$

Then $h_k \in \mathcal{B}_1^b(H(U))$ and the function $h := \lim_k h_k$ is a uniform limit of the functions $h_k$ on $\overline{U}$. Clearly, $h$ is strictly positive on $U \cap Z$ and is equal to zero on $U \setminus Z$. In particular, $h$ is harmonic on no nonempty subset of $U$. According to Remark 3.10, $h \notin \mathcal{B}_1(H(U))$.

Another example, where even a continuous function with the described properties is obtained, is the following: Let $U$ and $\{f_k\}$ be as in Example 3.11. By Lemma 3.6, the function

$$f = \sum_{k=1}^{\infty} 2^{-k} f_k$$

does not belong to $\mathcal{B}_1^b(H(U))$, although the partial sums do belong there and converge uniformly to $f$. ∎

*Remark 3.14:* We know from Remark 3.12 that, if $U$ is the unit ball, then each function $f \in \mathcal{B}_1^b(\overline{U})$ which is harmonic on $U$ belongs to $\mathcal{B}_1^b(H(U))$. This property certainly holds for some more general domains, but not for all open sets. Let $U$ be now the set $V$ from Lemma 3.6. Then, by Lemma 3.6, the characteristic function of $V$ does not belong to $\mathcal{B}_1^b(H(U))$, although it is lower semicontinuous on $\overline{U}$ and harmonic on $U$.

Now, we present two examples concerning $\mathcal{B}_1^b(H(U)^+)$ where $H(U)^+$ is the set of all positive functions in $H(U)$. This convex cone differs very much from $\mathcal{B}_1^b(H(U))$. Using the Harnack inequality it is easily verified that each convergent sequence of positive harmonic functions on $U$ with a finite limit converges locally uniformly and thus all functions from $\mathcal{B}_1^b(H(U)^+)$ are harmonic in $U$.

Despite this we observe again a pathological behavior of the class $\mathcal{B}_1^b(H(U)^+)$. We will show that $\mathcal{B}_1^b(H(U)^+)$ is not contained in $\mathcal{A}(H(U))$ and that $\mathcal{B}_1^b(H(U)^+)$ is not closed with respect to uniform convergence. Note that Fatou's lemma implies that all functions from $\mathcal{B}_1^b(H(U)^+)$ are $H(U)$-convex.

LEMMA 3.15: *Let $K$ be a metrizable compact space, $F$ a nonempty closed nowhere dense subset of $K$ and $\nu$ a positive Radon measure on $K$ with* $\operatorname{spt}\nu = K$. *Then there exists a sequence $\{f_n\}$ of positive continuous functions on $K$ such that $f_n \to 0$ on $K$ and $f_n\nu \overset{w^*}{\to} \chi_F\nu$* as $n \to \infty$.

*Proof:* Let $\rho$ be a metric compatible with the topology on $K$ and $g$ be the $\rho$-distance function from $F$. Fix a decreasing sequence $\{G_n\}$ of open sets with $\bigcap_{n=1}^\infty G_n = F$. For each $n \in \mathbb{N}$ we consider a "partition of unity" $\{\omega_{n,i}\}_{i=1}^{k(n)}$ such that each $\omega_{n,i}$ is a positive continuous function on $K$, $\operatorname{spt}\omega_{n,i} \subset G_n$, the $\rho$-diameter of $\operatorname{spt}\omega_{n,i}$ is less than $1/n$ and

$$\sum_{i=1}^{k(n)} \omega_{n,i} = 1 \quad \text{on } F.$$

For each $n \in \mathbb{N}$ and $i \in \{1, \dots, k(n)\}$ we find $\alpha_{n,i} \geq 0$ such that

$$\alpha_{n,i} \int_K g\omega_{n,i} d\nu = \int_F \omega_{n,i} d\nu.$$

(Here we have used the assumption that $F$ is nowhere dense.) Set

$$f_n = \sum_{i=1}^{k(n)} \alpha_{n,i} g\omega_{n,i}.$$

Given $\varphi \in \mathcal{C}(K)$ and $\varepsilon > 0$, by uniform continuity of $\varphi$ we find $n_0 \in \mathbb{N}$ such that

$$\operatorname{osc}_{\operatorname{spt}\omega_{n,i}} \varphi < \varepsilon, \quad n \geq n_0, i = 1, \dots, k(n).$$

Then, for $n \geq n_0$,

$$\left| \int_K f_n \varphi d\nu - \int_F \varphi d\nu \right| \leq \sum_{i=1}^{k(n)} \left| \int_K \alpha_{n,i} g \omega_{n,i} \varphi d\nu - \int_F \omega_{n,i} \varphi d\nu \right|$$
$$\leq \varepsilon \sum_{i=1}^{k(n)} \int_F \omega_{n,i} d\nu = \varepsilon \nu(F).$$

Hence $f_n \nu \xrightarrow{w^*} \chi_F \nu$ as $n \to \infty$. Obviously $f_n(x) = 0$ for every $n \in \mathbb{N}$ and every $x \in F$. If $x \in K \setminus F$, there exists $k \in \mathbb{N}$ such that $x \notin G_k$. Then $f_n(x) = 0$ whenever $n \geq k$. We conclude that $f_n \to 0$ on $K$ as $n \to \infty$. ∎

PROPOSITION 3.16: *Let $U$ be the open unit ball in $\mathbb{R}^m$ and $\sigma$ be the normalized surface measure on $\partial U$ so that $\sigma(\partial U) = 1$. Let $F \subset \partial U$ be a closed nowhere dense set with $\sigma(F) > 0$. Then the function*

$$f(x) := \begin{cases} \varepsilon_x^{\complement U}(\chi_F), & x \in U, \\ 0, & x \in \partial U, \end{cases}$$

*belongs to $\mathcal{B}_1^b(H(U)^+)$.*

*Proof:* By Lemma 3.15 there exists a sequence $\{f_n\}$ of positive continuous functions on $\partial U$ such that $f_n \to 0$ on $\partial U$ and $f_n \sigma \xrightarrow{w^*} \chi_F \sigma$ as $n \to \infty$. Recall that, for $x \in U$, the measure $\varepsilon_x^{\complement U}$ is absolutely continuous with respect to $\sigma$ and its density is the Poisson kernel

$$\frac{d\varepsilon_x^{\complement U}}{d\sigma}(y) = \frac{1 - |x|^2}{|y - x|^m}, \quad y \in \partial U.$$

The functions $h_n := f_n^{\complement U}$ belong to $H(U)^+$, $h_n = f_n$ on $\partial U$, so that $h_n \to 0$ on $\partial U$. If $x \in U$, then

$$h_n(x) = \int_{\partial U} \frac{1 - |x|^2}{|y - x|^m} f_n(y) d\sigma(y) \to \int_{\partial U} \frac{1 - |x|^2}{|y - x|^m} \chi_F(y) d\sigma(y) = \varepsilon_x^{\complement U}(\chi_F)$$

as $n \to \infty$. We see that $f \in \mathcal{B}_1(H(U)^+)$. ∎

*Remark 3.17:* It may be worthwhile to note that again we have constructed an example of a function from $\mathcal{B}_1^b(H(U))$ for which the "barycentric formula" does not hold. Namely, $f \in \mathcal{B}_1^b(H(U)) \setminus \mathcal{A}(H(U))$.

*Remark 3.18:* If $U$ is a bounded open subset of $\mathbb{R}^m$ and $0 \in U$, then the function $\chi_{\{0\}} \restriction \overline{U}$ from Example 3.7 is not in $\mathcal{B}_1(H(U)^+)$ because this would lead to a contradiction with the Harnack inequality for positive harmonic functions.

PROPOSITION 3.19: *Let $U$ be the open unit ball in $\mathbb{R}^m$ and $\sigma$ have the same meaning as in Proposition 3.16. Let $\{F_j\}$ be a sequence of pairwise disjoint nowhere dense closed subsets of $\partial U$ such that*

$$\sigma\left(\partial U \setminus \bigcup_{j=1}^{\infty} F_j\right) = 0.$$

*Let*

$$h_j(x) = \begin{cases} \varepsilon_x^{\complement U}(\chi_{F_j}), & x \in U, \\ 0, & x \in \partial U. \end{cases}$$

*Then the function*

$$h = \sum_{j=1}^{\infty} 2^{-j} h_j$$

*is a uniform limit of a sequence of functions from $\mathcal{B}_1^b(H(U)^+)$ and $h \in \mathcal{B}_1^b(H(U)) \setminus \mathcal{B}_1^b(H(U)^+)$. Even more: $h$ cannot be expressed as a pointwise limit of a lower bounded sequence of functions from $H(U)$.*

*Proof:* By Remark 3.12, $h \in \mathcal{B}_1^b(H(U))$. Supposing that $\{g_k\}$ is a lower bounded sequence of functions from $H(U)$ such that $g_k \to h$, we will deduce a contradiction. Then the sequence $\{g_k(0)\}$ is bounded and thus by the mean value property of harmonic functions

$$\sup_{k \in \mathbb{N}} \sigma(g_k) < \infty.$$

Taking into account that $\{g_k\}$ is lower bounded, we obtain

$$a := \sup_{k \in \mathbb{N}} \sigma(|g_k|) < \infty. \tag{6}$$

Let $G \neq \emptyset$ be a relatively open subset of $\partial U$. Then we can find $j \in \mathbb{N}$ and $z \in G \cap F_j$ such that $z$ is a point of density for $F_j$, i.e.,

$$\lim_{r \to 0} \frac{\sigma(B(z,r) \setminus F_j)}{r^{m-1}} = 0. \tag{7}$$

We fix $\rho \in (0,1)$ such that $\partial U \cap B(z,\rho) \subset G$. Let us suppose that

$$\sup\{|g_k(x)| \colon x \in \partial U \cap B(z,\rho),\ k \in \mathbb{N}\} < \infty. \tag{8}$$

We shall show that this assumption leads to a contradiction. By the Lebesgue dominated convergence theorem,

$$\varepsilon_k := \int_{\partial U\cap B(z,\rho)} |g_k| d\sigma \to \int_{\partial U\cap B(z,\rho)} \lim_k |g_k| d\sigma = 0.$$

It follows from (7) that we can fix $\delta \in (0, \rho/2)$ such that

$$a2^{m+1}\rho^{-m}\delta < 2^{-j-m}\delta^{1-m}\sigma(B(z,\delta)\cap F_j). \tag{9}$$

Set $x = (1-\delta)z$. Then $1-|x|^2 \le 2\delta$ and $|x-y| \ge \delta$ for all $y \in \partial U$. Hence

$$\begin{aligned}\int_{\partial U\cap B(z,\rho)} \frac{1-|x|^2}{|y-x|^m} g_k(y)d\sigma(y) &\le \frac{2\delta}{\delta^m}\int_{\partial U\cap B(z,\rho)} |g_k(y)|d\sigma(y)\\ &= 2\delta^{1-m}\varepsilon_k.\end{aligned} \tag{10}$$

Further, by (6)

$$\begin{aligned}\int_{\partial U\setminus B(z,\rho)} \frac{1-|x|^2}{|y-x|^m} g_k(y)d\sigma(y) &\le \frac{2\delta}{(\rho-\delta)^m}\int_{\partial U\setminus B(z,\rho)} |g_k(y)|d\sigma(y)\\ &\le \frac{2^{m+1}\delta}{\rho^m}\int_{\partial U} |g_k(y)|d\sigma(y)\\ &\le a2^{m+1}\rho^{-m}\delta.\end{aligned} \tag{11}$$

By (10) and (11) we have

$$\begin{aligned}g_k(x) &= \int_{\partial U\cap B(z,\rho)} \frac{1-|x|^2}{|y-x|^m} g_k(y)d\sigma(y) + \int_{\partial U\setminus B(z,\rho)} \frac{1-|x|^2}{|y-x|^m} g_k(y)d\sigma(y)\\ &\le 2\delta^{1-m}\varepsilon_k + a2^{m+1}\rho^{-m}\delta.\end{aligned}$$

Letting $k \to \infty$ we arrive at

$$h(x) \le a2^{m+1}\rho^{-m}\delta. \tag{12}$$

On the other hand,

$$\begin{aligned}h(x) &\ge 2^{-j}\int_{F_j\cap B(z,\delta)} \frac{1-|x|^2}{|y-x|^m} d\sigma(y) \ge 2^{-j}\frac{\delta}{(2\delta)^m}\sigma(F_j\cap B(z,\delta))\\ &\ge 2^{-j-m}\delta^{1-m}\sigma(B(z,\delta)\cap F_j).\end{aligned} \tag{13}$$

The inequalities (12) and (13) are not consistent with (9), and this contradiction disproves (8). We proved that for any nonempty relatively open subset $G$ of $\partial U$ there is a sequence $\{y_k\}$ in $G$ such that $\lim_k |g_k(y_k)| = \infty$. In particular, the set

$$G_k = \bigcup_{n\ge k} (\partial U \cap \{|g_n| > 1\})$$

is open and dense in $\partial U$. By the Baire category theorem we conclude that it cannot be true that $\{g_k\}$ converges to zero everywhere on $\partial U$. ∎

## 4. Affine functions on compact convex sets

Consider now a compact convex set $X$ in a locally convex space. Recall that $\mathfrak{A}(X)$ denotes the set of all affine functions on $X$ and $A(X) := \mathfrak{A}(X) \cap \mathcal{C}(X)$ is the set of all continuous affine functions on $X$. It is clear that $A(X)$ is a function space.

Given $\mu \in \mathcal{M}^1(X)$, there is a unique point $x_\mu \in X$, called the *barycenter* of $\mu$, such that

$$f(x_\mu) = \mu(f) \quad \text{for any } f \in A(X).$$

The next lemma characterizes affine continuous functions in the language of this approach. Its proof is straightforward and therefore we omit it. It can also be viewed as a simple illustration of Bauer's theorem, cf. Remark 5.6.

LEMMA 4.1: *Let $f$ be a continuous function on a compact convex set $X$. Then $f$ is affine if and only if $f$ is $A(X)$-affine. Shortly, $A(X) = \mathcal{A}^c(A(X))$.*

The fact that the barycentric formula characterizes affine continuous functions is just a special case of the following more general result. G. Choquet proved in [14] that the "barycentric formula" still holds for Baire-one affine functions.

Let $X$ be a compact convex subset of a locally convex space and $V \subset X$. For a real function $f$ on $X$ we denote

$$\begin{aligned} \operatorname{osc}_V f &= \sup\{|f(x) - f(y)| \colon x, y \in V\}, \\ \operatorname{osc}_V f(x) &= \inf\{\operatorname{osc}_{U \cap V} f \colon U \text{ is a neighborhood of } x\}, \\ \operatorname{osc} f(x) &= \operatorname{osc}_X f(x). \end{aligned}$$

By open we understand relatively open in $X$.

CHOQUET'S BARYCENTRIC THEOREM: *If $f$ is a Baire-one affine function on a compact convex set $X$ in a locally convex space, then $f$ is bounded and*

$$f(x) = \int_X f \, d\mu \quad \textit{for every } x \in X \textit{ and } \mu \in \mathcal{M}_x(A(X)).$$

*In other words, $\mathfrak{A}(X) \cap \mathcal{B}_1(X) \subset \mathcal{A}(A(X))$.*

*Proof:* See [1], Theorem I.2.6. We present here a slightly simplified proof. As $f$ is a Baire-one function, it has a point of continuity in $X$. Hence, $f$ is bounded on an open neighborhood of this point. Now, using compactness of $X$ and the affinity of $f$ we can simply check that $f$ is bounded on $X$.

Also, the proof of the barycentric formula follows the usual lines of Choquet's proof except for Lemma 4.2 below. Consider $x \in X$, $\mu \in \mathcal{M}_x(A(X))$ and choose $\varepsilon > 0$. By Lemma 4.2 below, there is a sequence $\{K_n\}$ of closed convex subsets of $X$ such that

$$\operatorname{osc}_{K_n} f < \varepsilon, \quad n \in \mathbb{N},$$
$$\mu\left(X \setminus \bigcup_{n=1}^{\infty} K_n\right) = 0.$$

We find $N \in \mathbb{N}$ such that

$$\mu(X \setminus (K_1 \cup \ldots \cup K_N)) < \varepsilon. \tag{14}$$

Denote

$$E_n = K_n \setminus \bigcup_{i<n} K_i, \quad n = 1, \ldots, N; \quad E_0 = X \setminus \bigcup_{i \le N} K_i,$$
$$\lambda_n = \mu(E_n), \quad n = 0, \ldots, N$$

and define probability Radon measures $\mu_n$, $n = 0, \ldots, N$, by

$$\mu_n = \begin{cases} \frac{1}{\lambda_n} \mu \restriction E_n & \text{if } \lambda_n > 0, \\ \varepsilon_x & \text{if } \lambda_n = 0. \end{cases}$$

Let $x_n$ be the barycenter of $\mu_n$, $n = 0, \ldots, N$. For $n \ge 1$ we have $x_n \in \overline{\text{co}} E_n \subset K_n$. Here $\overline{\text{co}} E$ denotes the closed convex hull of the set $E$. The oscillation properties of $f$ give

$$\begin{gathered} |\mu_n(f) - f(x_n)| < \varepsilon, \quad n = 1, \ldots, N, \\ |\mu_0(f) - f(x_0)| \le 2\|f\|. \end{gathered} \tag{15}$$

Obviously

$$\sum_{n=0}^{N} \lambda_n = 1, \quad \sum_{n=0}^{N} \lambda_n x_n = x, \quad \text{and} \quad \sum_{n=0}^{N} \lambda_n \mu_n = \mu. \tag{16}$$

Using the affinity of $f$, (14), (15) and (16) we have

$$\begin{aligned} |f(x) - \mu(f)| &= \left| f\Big(\sum_{n=0}^{N} \lambda_n x_n\Big) - \sum_{n=0}^{N} \lambda_n \mu_n(f) \right| = \left| \sum_{n=0}^{N} \lambda_n (f(x_n) - \mu_n(f)) \right| \\ &\le \lambda_0 |f(x_0) - \mu_0(f)| + \sum_{n=1}^{N} \lambda_n |f(x_n) - \mu_n(f)| \\ &\le \varepsilon(2\|f\| + 1). \end{aligned}$$

Letting $\varepsilon \to 0$ we conclude the proof. ∎

LEMMA 4.2: *Let $f$ be a Baire-one affine function on $X$ and $\varepsilon > 0$. Let $\mu$ be a probability Radon measure on $X$. Then there exists a countable collection $\{K_n\}$ of closed convex subsets of $X$ such that*

$$\begin{aligned} &\operatorname{osc}_{K_n} f < \varepsilon, \quad n \in \mathbb{N}, \\ &\mu\Big(X \setminus \bigcup_{n=1}^{\infty} K_n\Big) = 0. \end{aligned}$$

*Proof:* Fix $\varepsilon > 0$ and call (for the purpose of this proof) an open set $U \subset X$ *saturated* if there is a countable collection $\{K_n\}$ of closed convex subsets of $X$ such that

$$\begin{aligned} &\operatorname{osc}_{K_n} f < \varepsilon, \quad n \in \mathbb{N}, \\ &\mu\Big(U \setminus \bigcup_{n=1}^{\infty} K_n\Big) = 0. \end{aligned}$$

Clearly any countable union of saturated sets is saturated. But we prove more:

(17) *Any union of saturated sets is saturated.*

Indeed, if $V$ is a union of a collection of saturated sets, then $V$ is open and thus there is a sequence $\{H_k\}$ of compact sets such that $\mu(H_k) \nearrow \mu(V)$. Due to compactness, we can cover each $\{H_k\}$ by a finite family of saturated sets, and putting together all these families we obtain a countable family of saturated sets which covers $\mu$-almost all of $V$. It easily follows that $V$ itself is saturated, which proves (17).

Let $\mathcal{K}$ be the family of all closed convex subsets of $X$ whose complement in $X$ is saturated and $Z$ be the intersection of $\mathcal{K}$. By (17), $Z$ is the smallest element of $\mathcal{K}$. Set

$$Y = \{x \in Z\colon \operatorname{osc}_Z f(x) \geq \varepsilon\}.$$

Then $Y$ is a closed convex subset of $Z$. If $x \in Z \setminus Y$, then there is an open convex neighborhood $U$ of $x$ such that $\overline{U} \cap Y = \emptyset$ and $\operatorname{osc}_{\overline{U}\cap Z} f < \varepsilon$. Since $U \setminus Z$ is saturated and $\overline{U} \cap Z$ covers $U \cap Z$, we observe that $U$ is saturated. Appealing to (17), it follows that $Y$ is a closed convex subset of $Z$ whose complement in $X$ is saturated. By minimality of $Z$, we have $Y = Z$. There is no point of continuity of $f \restriction Z$, and since $f$ is Baire-one, it follows that $Z = \emptyset$. Hence $X$ is saturated, which concludes the proof. ∎

*Remark 4.3:* The Choquet barycentric theorem states that

$$\mathfrak{A}(X) \cap \mathcal{B}_1(X) \subset \mathcal{A}(A(X)) \cap \mathcal{B}_1(X).$$

The converse inclusion is also true. Indeed, given $x, y \in X$ and $\lambda \in (0,1)$, we have $\lambda\varepsilon_x + (1-\lambda)\varepsilon_y \in \mathcal{M}_{(\lambda x+(1-\lambda)y)}(A(X))$, and thus $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda)f(y)$ for any $f \in \mathcal{A}(A(X))$.

It seems to be interesting to state the following corollary to Choquet's barycentric theorem. As usual, $\operatorname{ext} X$ denotes the set of all extreme points of $X$.

$\mathcal{B}_1$-MAXIMUM PRINCIPLE: *Let $f$ be a Baire-one affine function on a compact convex set $X$ in a locally convex space. If $f \le 0$ on $\operatorname{ext} X$, then $f \le 0$ on $X$. Moreover,*

$$\sup\{|f(x)|: x \in X\} = \sup\{|f(x)|: x \in \operatorname{ext} X\}.$$

*Proof:* Fix an $x \in X$ and suppose that $f(x) > 0$. Since $f$ is a Baire-one function on $X$, the set $G := \{y \in X: f(y) \ge f(x)\}$ is a Baire $G_\delta$-set (recall that *Baire sets* are elements of the $\sigma$-algebra generated by zero sets of continuous functions), and due to the assumption we have $G \cap \operatorname{ext} X = \emptyset$. Let $\mu \in \mathcal{M}_x(A(X))$ be a maximal measure. According to the Bishop-de Leeuw theorem (see [9], cf. also [1], the remark subsequent to Corollary I.4.12), $\mu(G) = 0$. Applying Choquet's barycentric theorem we get

$$f(x) = \mu(f) = \int_{X \setminus G} f d\mu < \int_{X \setminus G} f(x) d\mu = f(x),$$

which is a contradiction. The second assertion is then an immediate consequence. ∎

*Remark 4.4:* There is a generalization of Choquet's barycentric theorem due to J. Saint Raymond [28]. He proved that *if $X$ is a compact convex subset of a locally convex space and $f$ is a Baire-one convex function on $X$, then*

$$f(x) \le \int_X f d\mu \quad \text{for every } x \in X \text{ and } \mu \in \mathcal{M}_x(A(X)).$$

Therefore, the $\mathcal{B}_1$-maximum principle can be strengthened and it continues to hold for the class of Baire-one convex functions.

*Remark 4.5:* A related result is the Bauer minimum principle concerning lower semicontinuous concave functions on compact convex sets, see [5]. A short proof based on the Krein–Milman theorem can be found in [30].

MOKOBODZKI'S APPROXIMATION THEOREM: *Let $X$ be a compact convex subset of a locally convex space and $f$ a Baire-one affine function on $X$. Then there exists a bounded sequence of continuous affine functions on $X$ pointwise converging to $f$ on $X$.*

*Proof* (Cf. Rogalski [29]): Let $f \in \mathcal{B}_1(X) \cap \mathfrak{A}(X)$. According to Choquet's barycentric theorem, $f$ is bounded. Since $f$ is a Baire-one function, there is a bounded increasing sequence $\{f_n\}$ of upper semicontinuous functions such that $f_n \nearrow f$ (see [24], 3.G.1). For any $x \in X$ and $n \in \mathbb{N}$, Lemma 2.1 guarantees the existence of a measure $\mu \in \mathcal{M}_x(A(X))$ such that $(f_n)^*(x) = \mu(f_n)$ (recall that $f^* = \inf\{h\colon h \in A(X), h \geq f\}$). Hence, using Choquet's barycentric theorem, $(f_n)^*(x) = \mu(f_n) \leq \mu(f) = f(x)$. Now, by an analogous reasoning, we can find a bounded decreasing sequence $\{g_n\}$ of lower semicontinuous functions such that $g_n \searrow f$ and $(g_n)_* \geq f$. Since $(f_n)^* - 1/n < f < (g_n)_* + 1/n$ for every $n$, an appeal to the Hahn–Banach theorem (cf. [15], Theorem 20.20) yields the existence of a function $h_n \in A(X)$ such that

$$f_n - 1/n \leq (f_n)^* - 1/n < h_n < (g_n)_* + 1/n \leq g_n + 1/n.$$

Then $\{h_n\}$ is a bounded sequence of continuous affine functions, $h_n \to f$ on $X$. ∎

*Remark 4.6:* We obtain the following chain of equalities:

$$\mathcal{A}(A(X)) \cap \mathcal{B}_1(X) = \mathfrak{A}(X) \cap \mathcal{B}_1(X) = \mathcal{B}_1^{bb}(A(X)) = \mathcal{B}_1^{b}(A(X)) = \mathcal{B}_1(A(X)).$$

Indeed, the first equality is just Remark 4.3. The inclusion $\mathfrak{A}(X) \cap \mathcal{B}_1(X) \subset \mathcal{B}_1^{bb}(A(X))$ is the Mokobodzki approximation theorem. Now, the remaining inclusions $\mathcal{B}_1^{bb}(A(X)) \subset \mathcal{B}_1^{b}(A(X)) \subset \mathcal{B}_1(A(X)) \subset \mathfrak{A}(X) \cap \mathcal{B}_1(X)$ are trivial.

*Remark 4.7:* Taking Mokobodzki's theorem for granted, Choquet's barycentric theorem is its immediate consequence.

## 5. A general criterion

We are going to characterize those bounded Baire-one functions which can be approximated pointwise by bounded sequences of elements of $\mathcal{H}$.

THEOREM 5.1: *Let $\mathcal{H}$ be a function space on a compact space $K$ and $f$ be a bounded Baire-one function on $K$. Then the following assertions are equivalent:*

(i) $f \in \mathcal{B}_1^{bb}(\mathcal{H})$,

(ii) *$f$ is completely $\mathcal{H}$-affine,*

(iii) *there exists $F \in \mathcal{B}_1^b(\mathbf{S}(\mathcal{H})) \cap \mathfrak{A}(\mathbf{S}(\mathcal{H}))$ such that $\mu(f) = F(\pi(\mu))$ for each $\mu \in \mathcal{M}^1(K)$,*

(iv) *there exists $F \in \mathcal{B}_1^b(\mathbf{S}(\mathcal{H})) \cap \mathfrak{A}(\mathbf{S}(\mathcal{H}))$ such that $f = F \circ \phi$,*

(v) *$f \in \mathcal{B}_1^{bb}(\overline{\mathcal{H}})$.*

*Proof:* The implication (i)$\Longrightarrow$(ii) follows immediately from the Lebesgue dominated convergence theorem.

To see that (ii)$\Longrightarrow$(iii), suppose $s \in \mathbf{S}(\mathcal{H})$ is given. By (4) there is $\mu_s \in \mathcal{M}^1(K)$ such that $\pi(\mu_s) = s$. Define $F(s) := \mu_s(f)$. This definition does not depend on the choice of $\mu_s$. Indeed, if $\pi(\mu_s) = \pi(\lambda_s)$, then $\mu_s - \lambda_s \in \mathcal{H}^\perp$. Thanks to the assumption (ii), $\mu_s(f) = \lambda_s(f)$. Obviously $F$ is an affine function on $\mathbf{S}(\mathcal{H})$ and satisfies $\mu(f) = F(\pi(\mu))$ for each $\mu \in \mathcal{M}^1(K)$. To show that $F$ is a Baire-one function, it is enough to show that $F^{-1}(U)$ is an $F_\sigma$-set whenever $U$ is an open subset of $\mathbb{R}$ (cf. [24], 3.A.1). So take an open set $U \subset \mathbb{R}$. Then

$$\begin{aligned} F^{-1}(U) &= \{\pi(\mu) \colon \mu \in \mathcal{M}^1(K), F(\pi(\mu)) \in U\} \\ &= \pi(\{\mu \in \mathcal{M}^1(K) \colon \mu(f) \in U\}). \end{aligned}$$

In a moment we shall show that the function $g\colon \mu \mapsto \mu(f)$ is a Baire-one function on $(\mathcal{M}^1(K), w^*)$. Granting this it will follow that $g^{-1}(U)$ is an $F_\sigma$-subset of the $w^*$-compact set $\mathcal{M}^1(K)$ and since $\pi$ is continuous, we get that $F^{-1}(U)$ is an $F_\sigma$-set as well. In order to prove that $g$ is a Baire-one function, suppose that $\{f_n\}$ is a bounded sequence in $\mathcal{C}(K)$, $f_n \to f$ on $K$. The functions $g_n\colon \mu \mapsto \mu(f_n)$ are continuous on $(\mathcal{M}^1(K), w^*)$, form a bounded sequence and $g_n \to g$.

For the proof that (iii)$\Longrightarrow$(iv), let $x \in K$ and $s_x = \phi(x) \in \mathbf{S}(\mathcal{H})$. Then $s_x = \pi(\varepsilon_x)$ and thus $F(s_x) = \varepsilon_x(f) = f(x)$.

Suppose now that (iv) holds. An appeal to Mokobodzki's theorem yields the existence of a bounded sequence $\{F_n\}$ of affine continuous functions on $\mathbf{S}(\mathcal{H})$ such that $F_n \to F$ on $\mathbf{S}(\mathcal{H})$. Let us define $f_n(x) = F_n(\phi(x))$. Then $f_n \to f$. By (5), $f_n \in \overline{\mathcal{H}}$. Therefore (iv)$\Longrightarrow$(v).

It is easy to see that (v)$\Longrightarrow$(i), so the theorem is proved. ∎

THEOREM 5.2: *Let $\mathcal{H}$ be a function space on a compact space $K$. Then there is an isometric isomorphism $T$ of $\mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{H})$ onto the space $\mathcal{B}_1^b(\mathbf{S}(\mathcal{H})) \cap \mathfrak{A}(\mathbf{S}(\mathcal{H}))$ such that $Tf(\pi(\mu)) = \mu(f)$ for any $f \in \mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{H})$ and $\mu \in \mathcal{M}^1(K)$. Further, $T = \Phi$ on $\mathcal{H}$.*

*Proof:* By Theorem 5.1, (ii)$\Longrightarrow$ (iii), for each $f \in \mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{H})$ there is $Tf \in \mathcal{B}_1^b(\mathbf{S}(\mathcal{H})) \cap \mathfrak{A}((\mathbf{S}(\mathcal{H}))$ such that $Tf(\pi(\mu)) = \mu(f)$ for each $\mu \in \mathcal{M}^1(K)$.

If $F \in \mathcal{B}_1^b(\mathbf{S}(\mathcal{H})) \cap \mathfrak{A}(\mathbf{S}(\mathcal{H}))$, then for the function $f = F \circ \phi$ we have $Tf = F$ (see Theorem 5.1, the proof of (iii)$\Longrightarrow$(iv)). On the other hand, each $f$ such that $Tf = F$ must be the function $F \circ \phi$ because

$$f(x) = \varepsilon_x(f) = Tf(\pi(\varepsilon_x)) = F(\phi(x)), \quad x \in K.$$

Hence $T$ is a bijection. Since the inverse mapping $F \mapsto F \circ \phi$ is linear, $T$ is linear as well. Given $f \in \mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{H})$, using (4) we obtain

$$\begin{aligned}\|Tf\| &= \sup_{s \in \mathbf{S}(\mathcal{H})} |Tf(s)| = \sup_{\mu \in \mathcal{M}^1(K)} |Tf(\pi(\mu))| \\ &= \sup_{\mu \in \mathcal{M}^1(K)} |\mu(f)| = \sup_{x \in K} |f(x)| \\ &= \|f\|.\end{aligned}$$

Choose now $h \in \mathcal{H}$ and $s \in \mathbf{S}(\mathcal{H})$. Then there is $\mu \in \mathcal{M}^1(K)$ such that $s = \pi(\mu)$ and

$$Th(s) = Th(\pi(\mu)) = \mu(h) = s(h) = \Phi(h)(s),$$

so $T = \Phi$ on $\mathcal{H}$. ∎

LEMMA 5.3: *Let $\mathcal{H}$ be a function space on a compact space $K$, $Z := (\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}))^\perp$. Then*

$$(\mathcal{A}^c(\mathcal{H}))^\perp = \overline{Z}^{w^*}.$$

*Proof:* Since $\mathcal{A}^c(\mathcal{H}) \subset \mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})$, we have $Z \subset (\mathcal{A}^c(\mathcal{H}))^\perp$. This annihilator is $w^*$-closed, so we get $\overline{Z}^{w^*} \subset (\mathcal{A}^c(\mathcal{H}))^\perp$.

Let $Y$ be the linear span of $\{\mathcal{M}_x(\mathcal{H}) - \mathcal{M}_x(\mathcal{H})\colon x \in K\}$. Then obviously $Y \subset Z$. Hence

$$\overline{Y}^{w^*} \subset \overline{Z}^{w^*} \subset \mathcal{A}^c(\mathcal{H}))^\perp. \tag{18}$$

On the other hand,

$$(\mathcal{A}^c(\mathcal{H}))^\perp \subset \overline{Y}^{w^*}.$$

Indeed, suppose $\mu \notin \overline{Y}^{w^*}$. There is $f \in \mathcal{C}(K)$ such that $\mu(f) = 1$ and $\lambda(f) = 0$ for any $\lambda \in \overline{Y}^{w^*}$. If $x \in K$ and $\nu \in \mathcal{M}_x(\mathcal{H})$, then $\nu - \varepsilon_x \in Y$ and therefore $\nu(f) = f(x)$. We see that $f$ is an $\mathcal{H}$-affine function and $\mu(f) \neq 0$. Hence $\mu \notin (\mathcal{A}^c(\mathcal{H}))^\perp$. ∎

Now, we are going to characterize those function spaces $\mathcal{H}$ for which bounded Baire-one $\mathcal{H}$-affine functions can be pointwise approximated by bounded sequences of $\mathcal{H}$-affine continuous functions.

THEOREM 5.4: *Let $\mathcal{H}$ be a function space on a compact space $K$. The following conditions are equivalent:*

(i) *for any function $f \in \mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})$ there exists a bounded sequence $\{h_n\}$ of $\mathcal{H}$-affine continuous functions such that $h_n \to f$ on $K$, i.e., $\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}) = \mathcal{B}_1^{bb}(\mathcal{A}^c(\mathcal{H}))$,*

(ii) $(\mathcal{A}^c(\mathcal{H}))^\perp = (\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}))^\perp$,

(iii) $(\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}))^\perp$ *is a $w^*$-closed subset of $\mathcal{M}(K)$,*

(iv) $\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}) = \mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{H})$,

(v) *there is an isometric isomorphism $T$ of $\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})$ onto the space of all affine Baire-one functions on $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ such that $T = \Phi$ on $\mathcal{A}^c(\mathcal{H})$ and $(Tf) \circ \phi = f$ for any $f \in \mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})$.*

*Proof:* It follows easily from Lemma 5.3 that $(\mathcal{A}^c(\mathcal{H}))^\perp = (\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}))^\perp$ if and only if the set $(\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H}))^\perp$ is a $w^*$-closed subset of $\mathcal{M}(K)$. Therefore, conditions (ii) and (iii) are equivalent. The implication (i)$\Longrightarrow$(ii) follows immediately from the Lebesgue dominated convergence theorem (cf. Theorem 5.1, (i)$\Longrightarrow$(ii)). The condition (iv) is just (ii) rewritten using another notation. The implication (iv)$\Longrightarrow$(v) follows from Theorem 5.2 and (v)$\Longrightarrow$(i) is a consequence of Theorem 5.1, (iv)$\Longrightarrow$(i). ∎

EXAMPLE 5.5: *There is a function space $\mathcal{H}$ on a (metrizable) compact set $K$ such that $(\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{H})) \setminus \mathcal{B}_1(\mathcal{H}) \neq \emptyset$.*

*Proof:* Let $r_n$, $n \in \mathbb{N}$ be mutually distinct points of $(0, \frac{1}{3})$ converging to $r_0 := 0$. Set

$$K_n = \{3r_n, 1 + r_n, 2 + r_n, 4 - r_n, 5 - r_n\}, \quad n = 0, 1, 2, \ldots,$$
$$K = \bigcup_{n=0}^{\infty} K_n.$$

Let $\mathcal{H}$ be the space of all continuous functions $f$ on $K$ satisfying the following conditions:

$$\begin{aligned} f(3r_n) &= (1 - r_n) f(0) + \frac{r_n}{2}(f(1 + r_n) + f(5 - r_n)) \\ &= (1 - r_n) f(0) + \frac{r_n}{2}(f(2 + r_n) + f(4 - r_n)). \end{aligned}$$

It is easy to see that $\mathcal{H}$ is a function space on $K$ and that $\mathcal{A}^c(\mathcal{H}) = \mathcal{H}$. Since $f(x) = x$ belongs to $\mathcal{H}$, we observe that $\mathcal{H}$ separates points of $K$. Fix $n \in \mathbb{N}$.

Any $z \in K_n \cap [1,5]$ is exposed and thus belongs to the Choquet boundary: we check this using the functions

$$f(x)=\begin{cases}1, & x=1+r_n,\\ -1, & x=5-r_n,\\ 0 & \text{elsewhere,}\end{cases} \quad\text{or}\quad f(x)=\begin{cases}1, & x=2+r_n,\\ -1, & x=4-r_n,\\ 0 & \text{elsewhere.}\end{cases}$$

Also the points $1,2,4,5$ are exposed: we check this using the functions

$$f(x)=\begin{cases}x-3, & x\in[2,4]\cap K,\\ 0 & \text{elsewhere,}\end{cases} \quad\text{or}\quad f(x)=\begin{cases}x-3, & x\in[1,5]\cap K,\\ 0 & \text{elsewhere.}\end{cases}$$

The point 0 belongs also to the Choquet boundary as the function $f(x)=x$ exposes it. Given $n\in\mathbb{N}$, the function

$$f(x)=\begin{cases}0, & x\in K_n\\ x & \text{elsewhere}\end{cases}$$

shows that each representing measure for $3r_n$ has its support in $\{0\}\cup K_n$.

Now, we define a Baire-one function $h$ on $K$ by

$$h(x)=\begin{cases}1, & x=1,\\ 0 & \text{elsewhere.}\end{cases}$$

Then $h$ is $\mathcal{H}$-affine. Indeed, from the above consideration, it is enough to verify $h(z)=\mu(h)$ for each $z=3r_n$, $n\in\mathbb{N}$, and $\mu\in\mathcal{M}_z(\mathcal{H})$. Since we have shown that $\mu(\{1\})=0$, the affinity is verified. Next we show that $h$ does not belong to $\mathcal{B}_1(\mathcal{H})$. We observe that any $f\in\mathcal{H}$ satisfies

$$f(1+r_n)+f(5-r_n)=f(2+r_n)+f(4-r_n).$$

Passing to limit as $n\to\infty$ we obtain also

$$f(1)+f(5)=f(2)+f(4).$$

The last identity must be satisfied also by all functions from $\mathcal{B}_1(\mathcal{H})$, but it is not satisfied by the function $h$. ∎

*Remark 5.6:* Let $\mathcal{H}$ be a function space on $K$. It is of interest to know under what conditions the function spaces $\mathcal{H}$ and $\mathcal{A}^c(\mathcal{H})$ coincide. Sometimes it is quite simple to decide. We already know that $A(X)=\mathcal{A}^c(A(X))$, cf. Lemma 4.1. Also, the equality $H(U)=\mathcal{A}^c(H(U))$ can be easily verified. On the other hand, the function space $\mathcal{H}$ of all second degree polynomials on $[-1,1]$ serves as an example of a closed function space where $\mathcal{A}^c(\mathcal{H})\neq\mathcal{H}$, in fact $\mathcal{A}^c(\mathcal{H})=\mathcal{C}([-1,1])$.

In general, the situation may be quite complicated. We recall the following result of H. Bauer from [6]:

BAUER'S THEOREM: *Let $\mathcal{H}$ be a function space on $K$. Then $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ if and only if there is a min-stable closed set $\mathcal{W} \subset \mathcal{C}(K)$ such that $\mathcal{H} = \mathcal{W} \cap (-\mathcal{W})$.*

## 6. Simplicial function spaces

Consider again a function space $\mathcal{H}$ on a compact space $K$. Let $f$ be a bounded Borel measurable function on $K$. Define

$$H^f: x \mapsto \int_K f d\delta_x, \quad x \in K,$$

where $\delta_x$ denotes the (unique) maximal measure in $\mathcal{M}_x(\mathcal{H})$.

PROPOSITION 6.1: *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space $K$ and $f \in \mathcal{C}(K)$. Then $H^f \in \mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{A}^c(\mathcal{H}))$.*

*Proof:* Choose $g \in \mathcal{K}^c(\mathcal{H})$. Then $H^g(x) = \delta_x(g) = g^*(x)$ for any $x \in K$ by Lemma 2.3. We see that $H^g$ is an upper semicontinuous function on $K$. Since $K$ is supposed to be metrizable, $H^g$ is a Baire-one function on $K$. Let $\mu \in (\mathcal{A}^c(\mathcal{H}))^\perp$ be given. Let $\mu = \mu_1 - \mu_2$ where $\mu_1$ and $\mu_2$ are positive Radon measures on $K$. By hypothesis, $\mu_1(h) = \mu_2(h)$ for any $h \in \mathcal{A}^c(\mathcal{H})$. It is an easy consequence of the Edwards separation theorem that the set $\{h \in \mathcal{A}^c(\mathcal{H}) \colon h \geq g\}$ forms a lower directed family of functions whenever $g \in \mathcal{K}^c(\mathcal{H})$. Therefore

$$\begin{aligned}
\mu_1(H^g) &= \mu_1(g^*) = \mu_1(\inf\{h\colon h \in \mathcal{A}^c(\mathcal{H}), h \geq g\}) \\
&= \inf\{\mu_1(h)\colon h \in \mathcal{A}^c(\mathcal{H}), h \geq g\} \\
&= \inf\{\mu_2(h)\colon h \in \mathcal{A}^c(\mathcal{H}), h \geq g\} \\
&= \mu_2(\inf\{h\colon h \in \mathcal{A}^c(\mathcal{H}), h \geq g\}) = \mu_2(g^*) = \mu_2(H^g),
\end{aligned}$$

i.e., $\mu(H^g) = 0$. Since the space $\mathcal{B}_1(K)$ and the set $\{\varphi \in \mathcal{C}(K)\colon \mu(H^\varphi) = 0\}$ are closed under the uniform convergence and the space $\mathcal{K}^c(\mathcal{H}) - \mathcal{K}^c(\mathcal{H})$ is uniformly dense in $\mathcal{C}(K)$ (this follows readily from the lattice version of the Stone–Weierstrass approximation theorem), we see that $H^f$ is a completely $\mathcal{A}^c(\mathcal{H})$-affine Baire-one function for each $f \in \mathcal{C}(K)$. ∎

COROLLARY 6.2: *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space $K$ and $f$ be a bounded Borel function on $K$. Then $H^f$ is a Borel function and belongs to $\mathbf{A}(\mathcal{A}^c(\mathcal{H}))$.*

*Proof:* Let $\mathcal{F}$ be the family of all bounded Borel functions on $K$ such that $H^f$ is a Borel function and belongs to $\mathbf{A}(\mathcal{A}^c(\mathcal{H}))$. Then by Proposition 6.1, $\mathcal{F}$ contains all continuous functions on $K$. Obviously $\mathcal{F}$ is closed with respect to limits of bounded sequences. We conclude that $\mathcal{F}$ contains all bounded Borel functions. ∎

THEOREM 6.3: *If a function space $\mathcal{H}$ on $K$ is simplicial, then for any bounded Baire-one $\mathcal{H}$-affine function $f$ on $K$ there exists a bounded sequence $\{h_n\}$ of continuous $\mathcal{H}$-affine functions such that $h_n \to f$ on $K$.*

*Proof:* According to Theorem 5.4, it is enough to show that $\mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{A}^c(\mathcal{H})) = \mathcal{B}_1^b(K) \cap \mathbf{A}(\mathcal{A}^c(\mathcal{H}))$. If $f \in \mathcal{B}_1^b(K) \cap \mathcal{A}(\mathcal{A}^c(\mathcal{H}))$, then $f = H^f$ and, by Corollary 6.2, $f$ is completely $\mathcal{A}^c(\mathcal{H})$-affine. ∎

COROLLARY 6.4: *Let $U$ be a bounded open subset of $\mathbb{R}^m$. Any bounded Baire-one $H(U)$-affine function on $\overline{U}$ is a pointwise limit of a bounded sequence of functions from $H(U)$.*

*Proof:* According to Proposition 3.4, the function space $H(U)$ is simplicial and, for any $x \in \overline{U}$, the balayage $\varepsilon_x^{\complement U}$ is a (unique) maximal measure in $\mathcal{M}_x(H(U))$. Now, it remains to use Theorem 6.3 since $\mathcal{A}^c(H(U)) = H(U)$. ∎

*Remark 6.4:* Let us now fix $f \in \mathcal{C}(\partial U)$. Thanks to Proposition 6.1, we have $f^{\complement U} \in \mathcal{B}_1^b(\overline{U}) \cap \mathcal{A}(H(U))$ and Corollary 6.4 tells us that $f^{\complement U}$ is on $\overline{U}$ a pointwise limit of a bounded sequence of functions from $H(U)$. Hence, knowing that the function space $H(U)$ is simplicial, we get again that *given $f \in \mathcal{C}(\partial U)$, there exists a bounded sequence $\{h_n\}$ in $H(U)$ such that $h_n \to f^{\complement U}$ on $\overline{U}$.* In the terminology of [29], we have shown that the operator $f \mapsto f^{\complement U}$ is a *strong Lion operator.*

Combining Theorem 3.2 with Proposition 3.4, we arrive in the "harmonic case" at the following proposition: *There exists a sequence $\{T_n\}$ of positive continuous linear operators from $\mathcal{C}(\partial U)$ into $H(U)$ such that $T_n f(x) \to \varepsilon_x^{\complement U}(f)$ for any $f \in \mathcal{C}(\partial U)$ and $x \in \overline{U}$.* Observe that $\varepsilon_x^{\complement U}$ is the unique maximal measure representing a point $x$. Now we are ready to derive a more general result.

The essential tool in the proof of Theorem 6.6 is Lazar's selection theorem (cf. [23], Theorem 3.1). Let us recall a few definitions. Let $X$ be a convex subset of a locally convex space and $E$ be another locally convex space. Suppose that $\Gamma$ is a map from $X$ into $2^E$, the family of all subsets of $E$. We call the map $\Gamma$ *affine*, if $\Gamma(x)$ is a nonempty convex subset of $E$ for every $x \in X$ and

$$\lambda\Gamma(x) + (1 - \lambda)\Gamma(y) \subset \Gamma(\lambda x + (1 - \lambda y))$$

whenever $0 < \lambda < 1$ and $x, y \in X$. A map $\Gamma$ is said to be *lower semicontinuous*, if for any open set $U \subset E$ the set

$$\Gamma^{-1}(U) := \{x \in X\colon \Gamma(x) \cap U \neq \emptyset\}$$

is open in X.

LAZAR'S SELECTION THEOREM: *Let $E$ be a Fréchet space (completely metrizable locally convex space), $X$ a Choquet simplex, and $\Gamma\colon X \to 2^E$ an affine lower semicontinuous map such that $\Gamma(x)$ is closed for every $x \in X$. Then there exists an affine continuous function $\gamma\colon X \to E$ with $\gamma(x) \in \Gamma(x)$ for each $x \in X$.*

THEOREM 6.6: *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space $K$. Then there exists a sequence $\{\gamma_n\}$ of affine continuous mappings,*

$$\gamma_n\colon \mathbf{S}(\mathcal{A}^c(\mathcal{H})) \to \mathcal{M}^1(K),$$

*such that $\pi(\gamma_n(s)) \xrightarrow{w*} s$ for every $s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$.*

*Proof:* Let $\{h_k\}$ be a dense subset of $\mathcal{A}^c(\mathcal{H})$. For any $n$ define a map $\Gamma_n$ on $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ whose values are subsets of $\mathcal{M}^1(K)$ as

$$\Gamma_n(s) = \bigcap_{k=1}^{n} \{\mu \in \mathcal{M}^1(K)\colon |\mu(h_k) - s(h_k)| < 1/n\}, \quad s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H})).$$

By (4), for each $s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ there exists $\mu \in \mathcal{M}^1(K)$ such that $\pi(\mu) = s$, and then of course $\mu \in \Gamma_n(s)$. Hence all maps $\Gamma_n$ have nonempty values. It is straightforward to verify that $\Gamma_n$ are affine, and so also $\overline{\Gamma}_n$ are affine. Here $\overline{\Gamma}_n$ is the map which assigns to each $s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ the closure of $\Gamma_n(s)$ in $(\mathcal{M}^1(K), w^*)$.

We show that $\Gamma_n$ and $\overline{\Gamma}_n$ are lower semicontinuous. To this end, let $n \in \mathbb{N}$ and let $V$ be an open subset of $\mathcal{M}^1(K)$. If

$$s \in \Gamma_n^{-1}(V) = \{s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H}))\colon \Gamma_n(s) \cap V \neq \emptyset\},$$

then there exists a measure $\mu \in V$ such that

$$|\mu(h_k) - s(h_k)| < 1/n$$

for any $k = 1, \ldots, n$. There exists an open neighbourhood $U$ of $s$ such that for any $t \in U$ and $k = 1, \ldots, n$ we have

$$|\mu(h_k) - t(h_k)| < 1/n.$$

Hence $\mu \in \Gamma_n(t)$ and we see that the set $\Gamma_n^{-1}(V)$ is open. Since

$$\{s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H})): \Gamma_n(s) \cap V \neq \emptyset\} = \{s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H})): \overline{\Gamma}_n(s) \cap V \neq \emptyset\},$$

the map $\overline{\Gamma}_n$ is also lower semicontinuous. By [8], $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ is a Choquet simplex. Notice also that $\mathcal{M}^1(K)$ admits an affine and homeomorphic embedding into the Fréchet space $\mathbb{R}^{\mathbb{N}}$. According to Lazar's selection theorem, for any $n$ there exists a continuous affine mapping $\gamma_n: \mathbf{S}(\mathcal{A}^c(\mathcal{H})) \to \mathcal{M}^1(K)$ such that $\gamma_n(s) \in \overline{\Gamma}_n(s)$ for any $s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$. If $s \in \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$, then obviously $\gamma_n(s)(h_k) \to s(h_k)$ for each $h_k$. Since the set $\{h_k\}$ is dense in $\mathcal{A}^c(\mathcal{H})$, it immediately follows that $\gamma_n(s)(h) \to s(h)$ for any $h \in \mathcal{A}^c(\mathcal{H})$. This verifies that $\pi(\gamma_n(s)) \xrightarrow{w^*} s$. ∎

THEOREM 6.7: *Let $\mathcal{H}$ be a simplicial function space on a metrizable compact space $K$. Then there exists a sequence of positive linear continuous operators $T_n: \mathcal{C}(K) \to \mathcal{A}^c(\mathcal{H})$ such that $T_n(f)(x) \to \delta_x(f)$ for any $f \in \mathcal{C}(K)$ and any $x \in K$.*

*Proof:* As in Section 2, let $\phi$ be the evaluation mapping from $K$ into $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$. We may apply Theorem 6.6 to obtain a sequence of continuous affine mappings $\gamma_n: \mathbf{S}(A^c(\mathcal{H})) \to \mathcal{M}^1(K)$ such that $\pi(\gamma_n(s)) \xrightarrow{w^*} s$ for each $s \in \mathbf{S}(A^c(\mathcal{H}))$. If $n \in \mathbb{N}$ and $f \in \mathcal{C}(K)$, set

$$\begin{aligned} \tau_n(f)(s) &:= \gamma_n(s)(f), \quad s \in \mathbf{S}(A^c(\mathcal{H})), \\ T_n f(x) &:= \tau_n(f)(\phi(x)), \quad x \in K. \end{aligned}$$

Obviously $\{T_n\}$ is a sequence of positive linear continuous operators on $\mathcal{C}(K)$ with values in $\mathcal{C}(K)$. By (5), $T_n f \in \mathcal{A}^c(\mathcal{H})$ for each $f \in \mathcal{C}(K)$. Fix $x \in \mathrm{Ch}_{\mathcal{H}}(K)$. We claim that

$$\gamma_n(s_x) \xrightarrow{w^*} \varepsilon_x. \tag{19}$$

Otherwise we would have $\gamma_{n_k}(s_x) \xrightarrow{w^*} \mu \neq \varepsilon_x$ for a subsequence, in view of compactness of $\mathcal{M}^1(K)$. Then we would obtain

$$\pi(\mu) = \lim_k \pi(\gamma_{n_k}(s_x)) = s_x.$$

Since $\varepsilon_x$ is the only $\mathcal{H}$-representing measure for $x$, it would follow that $\mu = \varepsilon_x$. This contradiction proves (19). Given $f \in \mathcal{C}(K)$, we see that for any $x \in \mathrm{Ch}_{\mathcal{H}}(K)$,

$$T_n f(x) = \tau_n(f)(s_x) = \gamma_n(s_x)(f) \to \varepsilon_x(f) = f(x).$$

Now, let $x \in K$ be arbitrary. Since in the metrizable case maximal measures are carried by $\mathrm{Ch}_{\mathcal{H}}(K)$ (cf. [1], Corollary 1.5.17), the Lebesgue dominated convergence theorem assures that

$$T_n f(x) = \int_{\mathrm{Ch}_{\mathcal{H}}(K)} T_n f(y) d\delta_x(y) \to \int_{\mathrm{Ch}_{\mathcal{H}}(K)} f(y) d\delta_x(y) = \delta_x(f)$$

for any $f \in \mathcal{C}(K)$, which finishes the proof of the theorem. ∎

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